Joint IMD-WMO group fellowship Training On Numerical Weather Prediction
By
Meteorological Training Institute, India Meteorological Department (IMD), Pune

Atmospheric Boundary Layer (ABL) and its Parameterizations – Part II

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- Largest scales – size of the flow, independent of direct viscous effects
- Smallest scales – dominated by viscous effects
- **Largest scales carry more Turbulence Kinetic Energy (TKE)**
- **Smallest scales carry much less TKE**

2. Turbulence Equations: Mass, Momentum, Concentration

$$\tilde{a}(\mathbf{x}, t) = A(\mathbf{x}, t) + a(\mathbf{x}, t)$$

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2. Turbulence Equations: Mean Flow Kinetic Energy

$$\frac{\partial}{\partial t}\frac{U_iU_i}{2} = -\frac{\partial}{\partial x_j}\left(\frac{U_iU_iU_j}{2} + \overline{U_iu_iu_j} - \nu\frac{\partial}{\partial x_j}\frac{U_iU_i}{2}\right) - \frac{U_i}{\rho}\frac{\partial P}{\partial x_i}$$

$$-\nu\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j} + \overline{u_iu_j}\,S_{ij}\,.$$

$$\frac{\partial U_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) = S_{ij} + R_{ij}$$

$$\overline{u_iu_j}\frac{\partial U_i}{\partial x_j} = \overline{u_iu_j}\left(S_{ij} + R_{ij}\right) = \overline{u_iu_j}\,S_{ij}$$

Time rate of change = flux divergence + pressure gradient work – viscous dissipation – loss to turbulence

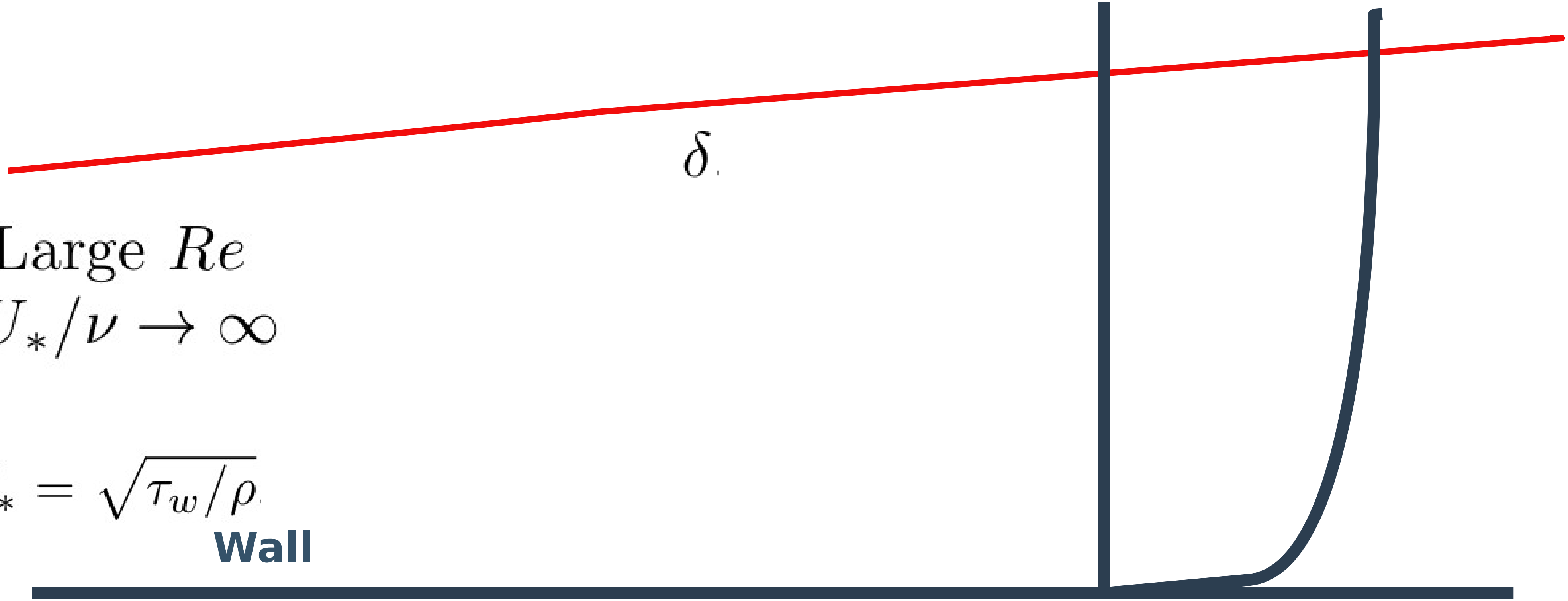
2. Turbulence Equations: TKE

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{u_i u_i} = -\frac{U_j}{2}\frac{\partial}{\partial x_j}\overline{u_i u_i} - \overline{u_i u_j}\frac{\partial U_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\overline{u_i u_i u_j}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{p u_i} - \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}.$$

Time rate of change = advective divergence + gain from mean flow + pressure and flux divergence + viscous dissipation

3. Turbulent Boundary Layer



Large Re

$\delta U_* / \nu \to \infty$

$U_* = \sqrt{\tau_w / \rho}$

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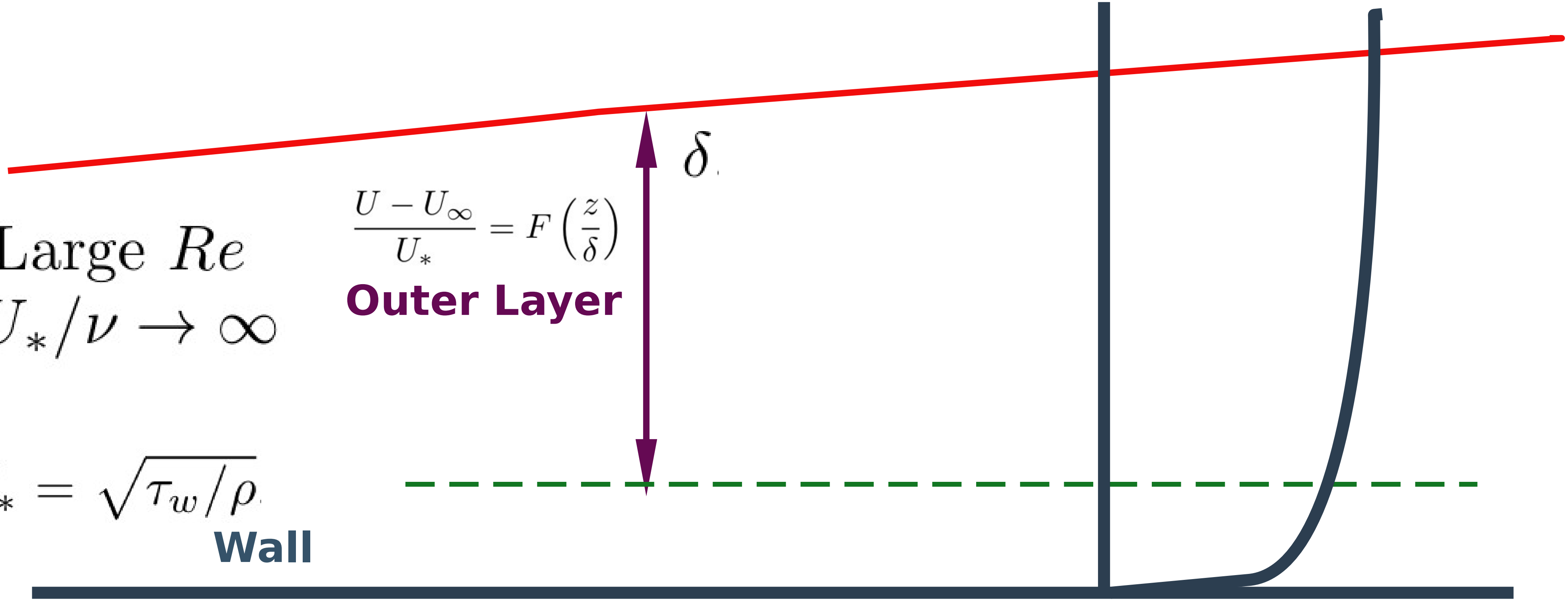

$$\frac{U - U_\infty}{U_*} = F\left(\frac{z}{\delta}\right)$$

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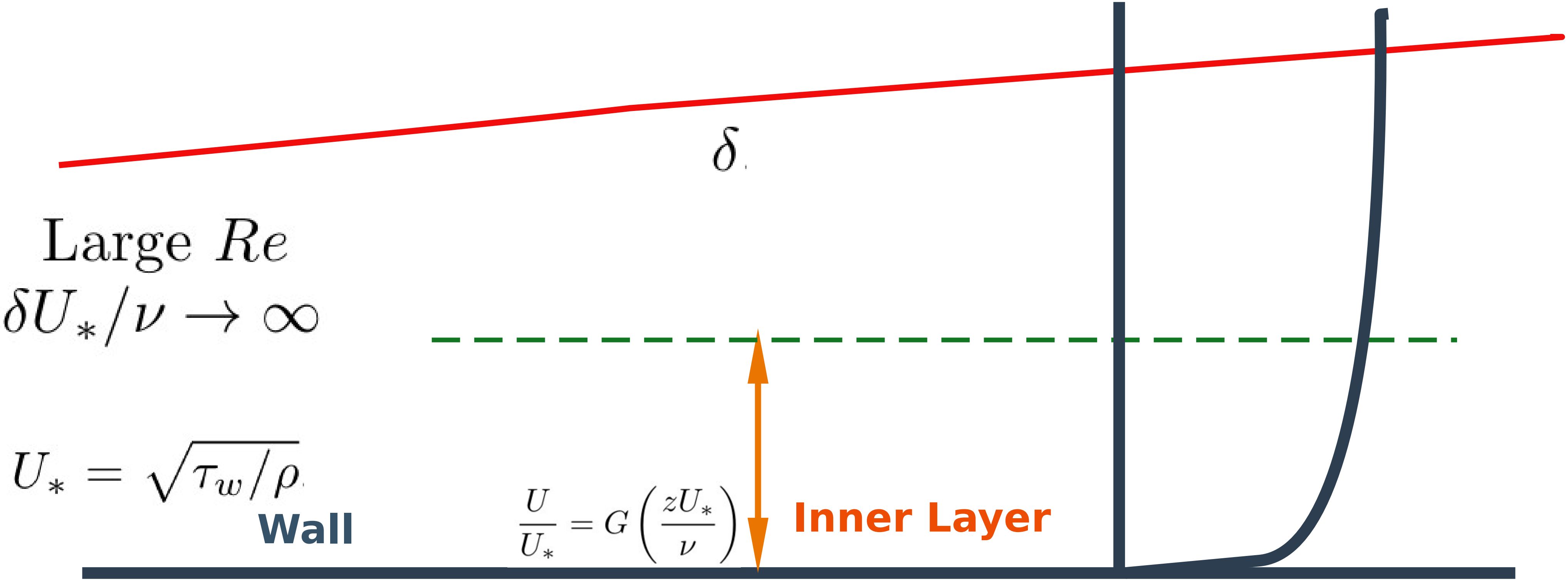

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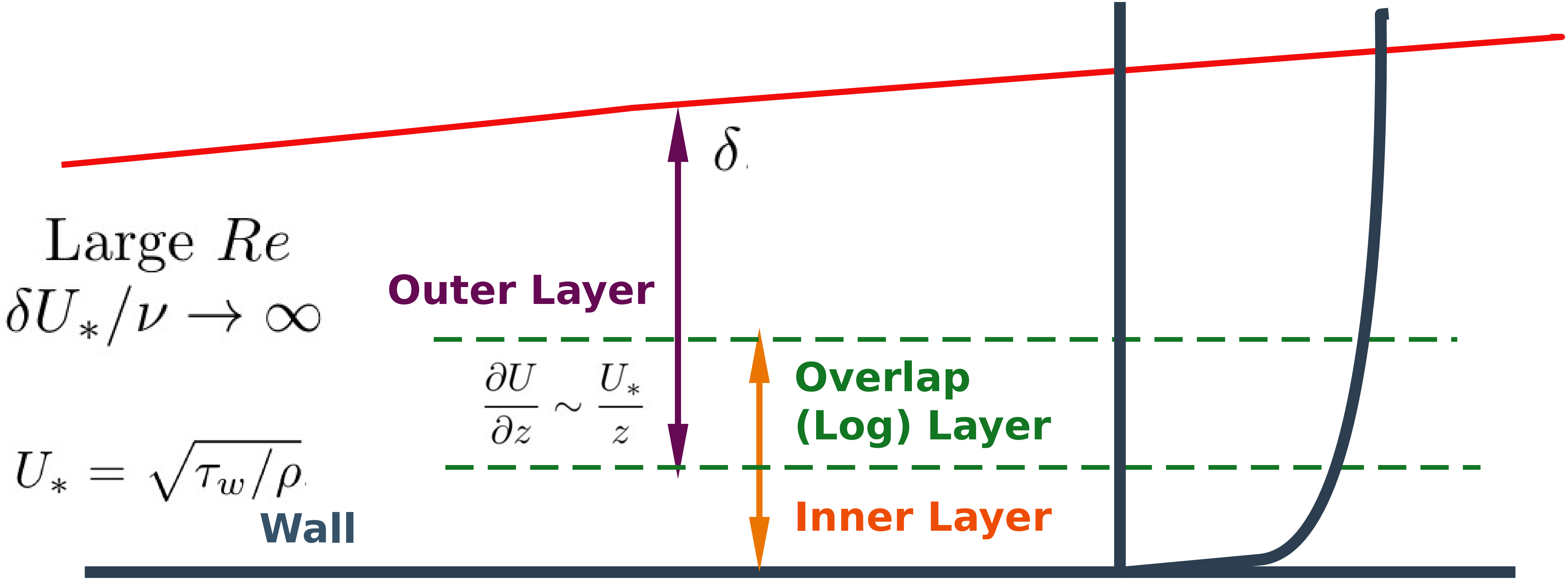

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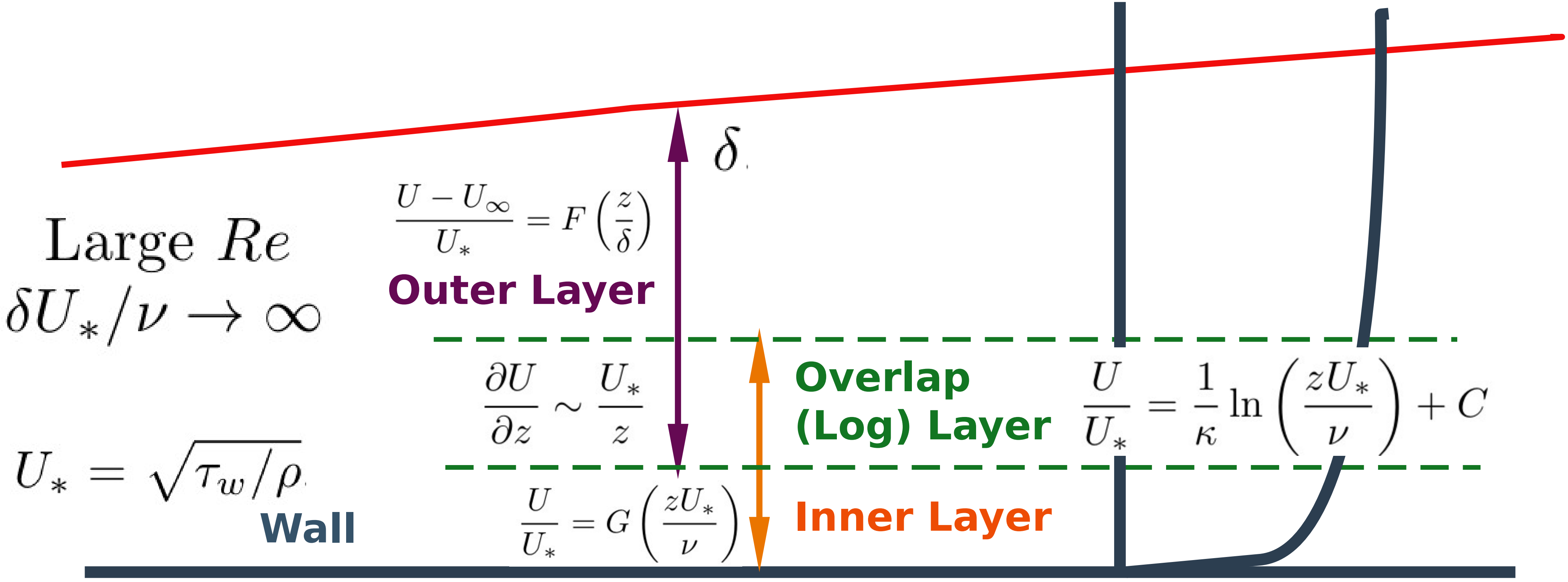

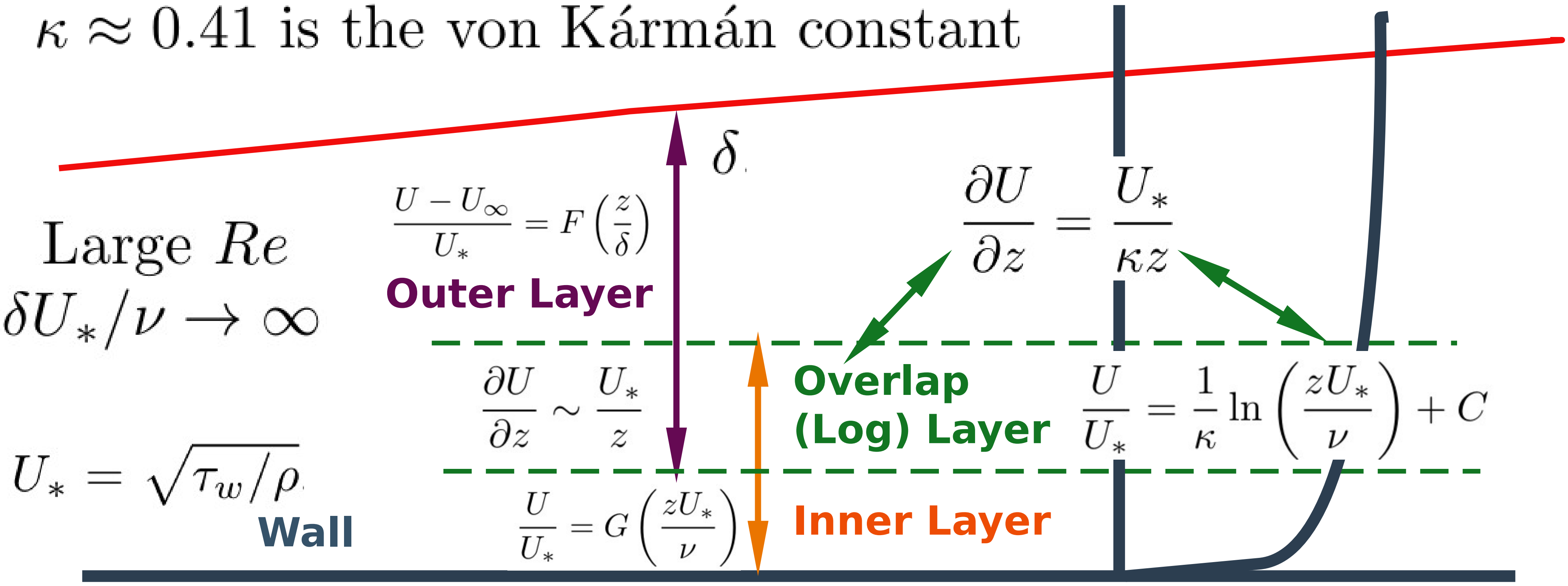
$\kappa \approx 0.41$ is the von Kármán constant
δ
Large Re
$\delta U_* / \nu \to \infty$
$U_* = \sqrt{\tau_w / \rho}$
Wall
$\frac{U - U_\infty}{U_*} = F\left(\frac{z}{\delta}\right)$
Outer Layer
$\frac{\partial U}{\partial z} \sim \frac{U_*}{z}$
$\frac{U}{U_*} = G\left(\frac{zU_*}{\nu}\right)$
$\frac{\partial U}{\partial z} = \frac{U_*}{\kappa z}$
Overlap
(Log) Layer
$\frac{U}{U_*} = \frac{1}{\kappa} \ln\left(\frac{zU_*}{\nu}\right) + C$
Inner Layer

4. Atmospheric Turbulence – Dry Air Equations

Hydrostatic "Background" OR "Base" State

$p_0 = \rho_0 R_d T_0$ **Eqn. of thermodynamic state**

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Isentropic, imaginary displacements of fluid parcel in the vertical direction

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$$\frac{dT_0}{dx_3} = -\frac{g}{c_p}$$

Base state vertical variation of enthalpy and temperature

4. Atmospheric Turbulence – Dry Air Equations

Flow-Induced “Small” Perturbations Around the “Base” State

$$\tilde{p} = p_0(z) + \tilde{p}'(\mathbf{x}, t); \qquad \tilde{T} = T_0(z) + \tilde{T}'(\mathbf{x}, t); \qquad \tilde{\rho} = \rho_0(z) + \tilde{\rho}'(\mathbf{x}, t)$$

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$$= -\frac{\rho_0}{T_0}\tilde{T}' + \frac{1}{R_d T_0}\tilde{p}'.$$

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Density fluctuation is proportional to temperature fluctuation

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Mass Conservation Eqn.

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$$\tilde{u}_3 \frac{d\rho_0}{dx_3} + \rho_0 \frac{\partial \tilde{u}_i}{\partial x_i} \simeq 0.$$

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$$\frac{\partial \tilde{u}_i}{\partial x_i} \simeq -\frac{\tilde{u}_3}{\rho_0}\frac{d\rho_0}{dx_3} = \frac{\tilde{u}_3}{H_\rho}$$

Mass Conservation Eqn.

If boundary layer height << Hp, then divergence can be assumed to be negligible

4. Atmospheric Turbulence – Dry Air Equations

Final set of dry-air Instantaneous Equations

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0$$

Mass Conservation

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Momentum Equation

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Momentum Equation

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u}_i \frac{\partial \tilde{\theta}}{\partial x_i} = \alpha \nabla^2 \tilde{\theta} - \frac{\tilde{\theta}}{\rho c_p \tilde{T}} \frac{\partial \tilde{R}_i}{\partial x_i}$$

Thermal Energy Equation

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Momentum Equation

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u}_i \frac{\partial \tilde{\theta}}{\partial x_i} = \boxed{\alpha \nabla^2 \tilde{\theta}} - \boxed{\frac{\tilde{\theta}}{\rho c_p \tilde{T}}\frac{\partial \tilde{R}_i}{\partial x_i}}$$

Thermal Energy Equation

$$\frac{\partial \tilde{c}}{\partial t} + \tilde{u}_i \frac{\partial \tilde{c}}{\partial x_i} = \boxed{\gamma \frac{\partial^2 \tilde{c}}{\partial x_i \partial x_i}}$$

“Passive” Scalar Conservation Equation

4. Atmospheric Turbulence – Moist Air Equations

Final set of moist-air Instantaneous Equations

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}'}{\partial x_i} - 2\epsilon_{ijk}\Omega_j \tilde{u}_k + \frac{g}{\theta_0}\tilde{\theta}'_v \delta_{3i} + \nu \nabla^2 \tilde{u}_i$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0,$$

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\theta}}{\partial x_j} = \alpha \nabla^2 \tilde{\theta} - \frac{\tilde{\theta}}{\rho_0 c_p \tilde{T}} \frac{\partial \tilde{R}_j}{\partial x_j},$$

$$\frac{\partial \tilde{q}}{\partial t} + \tilde{u}_i \frac{\partial \tilde{q}}{\partial x_i} = \gamma \frac{\partial^2 \tilde{q}}{\partial x_i \partial x_i}.$$

Final set of moist-air Averaged Equations

$$\tilde{u}_i = U_i + u_i, \qquad \tilde{p}' = P + p, \qquad \tilde{\theta}_{\mathrm{v}}' = \Theta_{\mathrm{v}}' + \theta_{\mathrm{v}},$$

$$\frac{\tilde{\theta}}{\rho_0 c_p \tilde{T}} \frac{\partial \tilde{R}_i}{\partial x_i} = \mathcal{R} + r, \qquad \tilde{c} = C + c, \qquad \tilde{\theta} = \Theta + \theta.$$

4. Atmospheric Turbulence – Moist Air Equations

Final set of moist-air Averaged Equations

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u_i u_j} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} - 2\epsilon_{ijk}\Omega_j U_k + \frac{g}{\theta_0}\Theta_{\mathrm{v}}'\,\delta_{3i}\,,$$

$$\frac{\partial U_i}{\partial x_i} = 0.$$

$$\frac{\partial \Theta}{\partial t} + U_i \frac{\partial \Theta}{\partial x_i} + \frac{\partial \overline{\theta u_i}}{\partial x_i} + \mathcal{R} = 0,$$

$$\frac{\partial C}{\partial t} + U_i \frac{\partial C}{\partial x_i} + \frac{\partial \overline{c u_i}}{\partial x_i} = 0.$$

Mean Flow

4. Atmospheric Turbulence – Moist Air Equations

Final set of moist-air Averaged Equations

$$\frac{\partial \overline{u_i u_k}}{\partial t} + U_j \frac{\partial \overline{u_i u_k}}{\partial x_j} =$$

$$- \overline{u_j u_k} \frac{\partial U_i}{\partial x_j} - \overline{u_j u_i} \frac{\partial U_k}{\partial x_j} \quad \text{(mean-gradient production)}$$

$$- \frac{\partial \overline{u_i u_k u_j}}{\partial x_j} \quad \text{(turbulent transport)}$$

$$- \frac{1}{\rho_0} \left(\overline{u_k \frac{\partial p}{\partial x_i}} + \overline{u_i \frac{\partial p}{\partial x_k}} \right) \quad \text{(pressure-gradient interaction)}$$

$$- 2\epsilon_{ijm} \Omega_j \overline{u_m u_k} - 2\epsilon_{kjm} \Omega_j \overline{u_m u_i} \quad \text{(Coriolis)}$$

$$+ \frac{g}{\theta_0} \left(\overline{\theta_v u_k} \delta_{3i} + \overline{\theta_v u_i} \delta_{3k} \right) \quad \text{(buoyant production)}$$

$$- \frac{2\epsilon}{3} \delta_{ik} \quad \text{(viscous dissipation)}.$$

Reynolds Stresses